Discrete Mathematics 312 (2012) 1218–1221

Contents lists available at SciVerse ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Note

Complements and consistent closures

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ARTICLE INFO

Article history:
Received 24 February 2011
Received in revised form 9 August 2011
Accepted 28 November 2011
Available online 3 January 2012

Keywords:
Closure
Consistency
Complement
Closure-complement problem

ABSTRACT

We prove that the complement of the consistent closure of the complement of the consistent closure is consistent, and thus, at most 10 binary relations can be generated from a binary relation by taking complements and consistent closures.

1. Introduction

The present paper is concerned with the well-known closure-complement problem. Kuratowski [16] is a pioneering work on this subject. He showed that in a topological space at most 14 distinct sets can be generated from a set A by taking complements and closures. Graham et al. [14] was the first work to establish a version of the closure-complement theorem for operators on binary relations. They show that the complement of the transitive closure of the complement of the transitive closure is transitive. Since complementation is involutive and the transitive closure is idempotent, a series of binary relations is obtained by alternating applications of the two operators. Thus, at most 10 binary relations can be generated from a binary relation R by taking complements and transitive closures. Fishburn [12,13] and Peleg [17] proved related results for operations on binary relations.

In the present paper, we aim to extend the result of [14]. Instead of transitivity, we focus on another coherence property of binary relations, *consistency*, which is a weakening of transitivity. Consistency was introduced by Suzumura [19], and it requires that for all $K \geq 1$, if

$$(x^{k-1}, x^k) \in R \quad \text{for all } k \in \{1, \ldots, K\},$$

then

$$(x^K, x^0) \notin \{(x, y) \in X \times X : (x, y) \in R \text{ and } (y, x) \notin R\},$$

where R is a binary relation. Recently, Bossert et al. [5] have introduced the concept of *consistent closure*: for every binary relation, its consistent closure gives the smallest consistent binary relation containing it.[1] We show that the complement of the consistent closure of the complement of the consistent closure is consistent: thus, at most 10 binary relations can be generated from R by taking complements and consistent closures because the consistent closure is idempotent. This result is analogous to the result of [14] on transitive closures.

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[1] See also Bossert and Suzumura [7].

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doi:10.1016/j.disc.2011.11.033

We explain why consistency is an important property. The first reason is related to Szpilrajn's extension theorem. Szpilrajn [21] showed that every quasi-ordering (transitive and reflexive binary relation) has an ordering extension.[2] Suzumura [19] showed that consistency is a necessary and sufficient condition for the existence of an ordering extension.[3] The second reason is related to an interpretation of binary relations. In economic theory, each consumer is required to have a binary relation R on the set of alternatives—a *preference* relation. We read $(x, y) \in R$ as "x is at least as good as y" (thus $(x, y) \in \{(x, y) \in X \times X : (x, y) \in R \text{ and } (y, x) \notin R\}$ means "x is preferred to y"). The work by Bossert and Suzumura [6] includes an application of consistency to consumer theory. A requirement of consistency is closely related to the problem of a "money pump" [9], [18, p.78], [22]. Suppose that a consumer has the following preference relation:

$$(x, y) \in R \,\&\, (y, z) \in R \,\&\, (z, x) \in R \,\&\, (x, z) \notin R.$$

This preference violates consistency. Since z is strictly preferred to x, he/she is willing to pay some amount of money to exchange x for z. Moreover, he/she voluntarily exchanges z for y and y for x. Then, the consumer ends up with the original alternative but with less money. As such, a "money pump" occurs when consistency is violated. The third reason is related to a numerical representation of a binary relation. As shown by Andrikopoulos [1], consistency is a necessary and sufficient condition for the existence of a weak representation of a binary relation in a countable set. Thus, consistency is significant for the existence of the consumer's utility function.

2. Preliminaries

The set of alternatives is X with $|X| \geq 2$. Let $R \subseteq X \times X$ be a binary relation on X. The symmetric and the asymmetric part of R are denoted by $I(R)$ and $P(R)$, respectively. These are defined by $I(R) = \{(x, y) \in X \times X : (x, y) \in R \text{ and } (y, x) \in R\}$ and $P(R) = \{(x, y) \in X \times X : (x, y) \in R \text{ and } (y, x) \notin R\}$. The *diagonal relation* on X is given by $\Delta = \{(x, y) \in X \times X : x = y\}$.

R is *transitive* if and only if, for all $x, y, z \in X$,

$$[(x, y) \in R \text{ and } (y, z) \in R] \Rightarrow (x, z) \in R,$$

and R is *consistent* if and only if, for all $K \geq 1$ and all $x^0, \ldots, x^K \in X$,

$$[(x^{k-1}, x^k) \in R \;\forall k \in \{1, \ldots, K\}] \Rightarrow (x^K, x^0) \notin P(R).$$

Transitivity implies consistency, but not vice versa.

The *composition* of two binary relations R and R' on X is defined by

$$R \circ R' = \{(x, z) \in X \times X : (x, y) \in R \text{ and } (y, z) \in R' \text{ for some } y \in X\}.$$

Define a sequence of binary relations $\{R^{(\tau)}\}_{\tau=0}^{\infty}$ by

$$R^0 = R \quad \text{and} \quad R^{(\tau)} = R^{(\tau-1)} \circ R \quad \text{for } \tau \in \mathbb{N}.$$

Let R^t be the *transitive closure* of R, i.e.,

$$R^t = \bigcup_{\tau=0}^{\infty} R^{(\tau)}.$$

Note that R^t is the smallest transitive binary relation containing R. Moreover, for any two binary relations R, T on X, if $R \subseteq T$, then $R^t \subseteq T^t$.

Recently, the concept of *consistent closure* was proposed by Bossert et al. [5]. For any binary relation, the consistent closure of R, denoted as R^s, is defined as follows:

$$R^s = R \cup \{(x, y) \in X \times X : (y, x) \in R \text{ and } (x, y) \in R^t\}.$$

Bossert et al. [5] showed that for any binary relation R on X, R^s is the smallest consistent binary relation containing R. By definition, $R \subseteq R^s$ for any binary relation R on X.

The *complement* of R, denoted by R^-, is defined by

$$R^- = X \times X - R.$$

For simplicity of notation, we write R^{s-}, R^{-s}, R^{s-s}, etc., instead of $(R^s)^-$, $(R^-)^s$, $((R^s)^-)^s$, etc.

We now provide an example.

Example 1. Assume that $X = \{x, y, z\}$, $R = \{(x, y), (y, z)\}$, and $Q = \{(x, y), (y, z), (x, z), (z, x)\}$. R is consistent but not transitive, and Q and $Q \cup \Delta$ are not consistent (thus are not transitive). Thus, $R^s = R$ and $R^t = R \cup \{(x, z)\}$, while $Q^s \neq Q^t$ and $(Q \cup \Delta)^s = (Q \cup \Delta)^t$.

3. Results

We now state our theorem.

[2] The original theorem of Szpilrajn [21] states that every partial order has a linear extension. The "quasi-ordering" version is stated by Arrow [2] without proof. Its formal proof was provided by Hansson [15]. Dushnik and Miller [11], Donaldson and Weymark [10], Bossert [4], Bosi and Herden [3], and Cato [8] proved related results.

[3] See also Suzumura [20].

Theorem 1. *For any binary relation $R \subseteq X \times X$, $R^{s-s-s} = R^{s-s-}$. Therefore, at most 10 binary relations can be generated from R by taking complements and consistent closures, namely, R, R^s, R^-, R^{s-}, R^{-s}, R^{s-s}, R^{-s-}, R^{s-s-}, R^{-s-s}, R^{-s-s-}.*

It is worth noting that in general, analogous results do not hold for closure operators: for example, the complement of the reflexive closure of the complement of the reflexive closure is not reflexive.

To prove Theorem 1, we establish two lemmas.

Lemma 1. $R^{-s-} \subseteq R$.

Proof. For $R, T \subseteq X \times X$, if $R \subseteq T$, then $T^- \subseteq R^-$. Since $R^- \subseteq R^{-s}$, it follows that $R^{-s-} \subseteq R$. □

Lemma 2. *For $\tau \in \mathbb{N}$, if $(x, y) \in (R^s)^{(\tau)} \circ R^{s-s-}$, then $(y, x) \notin P(R^{s-s-})$.*

Proof. On the contrary, suppose that there exists $\tau \in \mathbb{N}$ such that $(x, y) \in (R^s)^{(\tau)} \circ R^{s-s-}$ and $(y, x) \in P(R^{s-s-})$. Lemma 1 implies that $R^{s-s-} \subseteq R^s$, so $(x, y) \in (R^s)^{(\tau)} \circ R^s$. Since R^s is consistent,

$$(x, y) \in (R^s)^{(\tau)} \circ R^s \Rightarrow (y, x) \notin P(R^s).$$

Note that $(y, x) \in P(R^{s-s-})$ if and only if

$$(y, x) \in R^{s-s-} \tag{1}$$

and

$$(x, y) \in R^{s-s}. \tag{2}$$

Since $R^{s-s-} \subseteq R^s$ by Lemma 1, (1) implies that

$$(y, x) \in R^s. \tag{3}$$

Since $(y, x) \notin P(R^s)$ and $(y, x) \in R^s$, we have $(x, y) \in R^s$. Thus, it follows that

$$(x, y) \notin R^{s-}. \tag{4}$$

From (2) and (4), we obtain $(x, y) \in R^{s-s} - R^{s-}$. By definition of consistent closure, $(x, y) \in tc(R^{s-})$ and $(y, x) \in R^{s-}$. Hence, $(y, x) \notin R^s$, which contradicts (3). □

Proof of Theorem 1. The second part immediately follows from the first part; thus, we need only prove the first part. Take any $\tau \in \mathbb{N} \setminus \{1\}$. Suppose that $(x, y) \in (R^{s-s-})^{(\tau)}$. Since $R^{s-s-} \subseteq R^s$ by Lemma 1, $(x, y) \in (R^s)^{(\tau-1)} \circ R^{s-s-}$. Hence, Lemma 2 implies that $(y, x) \notin P(R^{s-s-})$. □

The reader may wonder whether $R^{s-s-} = R^{t-t-}$ for any R. The following example demonstrates that R^{s-s-} is not identical with R^{t-t-} in general.

Example 2. Assume that $X = \{x, y, z\}$ and $R = \{(x, y), (y, z)\}$. In this case,

$$\begin{array}{ll} R^s = R, & R^t = R \cup \{(x, z)\} \\ R^{s-} = \{(y, x), (z, y), (x, z), (z, x)\} \cup \Delta, & R^{t-} = \{(y, x), (z, y), (z, x)\} \cup \Delta \\ R^{s-s} = X \times X, & R^{t-t} = R^{t-} \\ R^{s-s-} = \emptyset, & R^{t-t-} = R^t. \end{array}$$

There are few papers that study consistent closure. Our result indicates that consistent closure might provide a fruitful field in the analysis of operators on binary relations. It remains for future research to examine the complete picture of binary relations when other operators, such as the asymmetric part, the dual, and the symmetric part, are added.

Acknowledgments

I am grateful to the Editor and two anonymous referees of this journal for valuable comments as well as Kotaro Suzumura for discussions over several years. This paper was financially supported by Grant-in-Aids for Young Scientists (B) from the Japanese Society for the Promotion of Science and the Ministry of Education, Culture, Sports, Science and Technology.

References

[1] A. Andrikopoulos, Characterization of the existence of semicontinuous weak utilities for binary relations, Theory and Decision 70 (2011) 13–26.
[2] K.J. Arrow, Social Choice and Individual Values, Wiley, New York, 1951.
[3] G. Bosi, G. Herden, On a possible continuous analogue of the Szpilrajn theorem and its strengthening by Dushnik and Miller, Order 23 (2006) 71–296.

[4] W. Bossert, Intersection quasi-orderings: an alternative proof, Order 16 (1999) 221–225.
[5] W. Bossert, Y. Sprumont, K. Suzumura, Consistent rationalizability, Economica 72 (2005) 185–200.
[6] W. Bossert, K. Suzumura, Domain closedness conditions and rational choice, Order 24 (2007) 75–88.
[7] W. Bossert, K. Suzumura, Consistency, Choice and Rationality, Harvard University Press, 2010.
[8] S. Cato, Szpilrajn, Arrow, and Suzumura: concise proofs of extension theorems and an extension, Metroeconomica (2011), forthcoming (doi:10.1111/j.1467-999X.2011.04130.x).
[9] D. Davidson, J.C.C. McKinsey, P. Suppes, Outlines of a formal theory of value, I, Philos. Sci. 22 (1955) 140–160.
[10] D. Donaldson, J.A. Weymark, A quasiordering is the intersection of orderings, J. Econom. Theory 78 (1998) 382–387.
[11] B. Dushnik, E. Miller, Partially ordered sets, Amer. J. Math. 63 (1941) 600–610.
[12] P.C. Fishburn, Operations on binary relations, Discrete Math. 21 (1978) 7–22.
[13] P.C. Fishburn, Transitivity, Rev. Econom. Stud. 46 (1979) 163–173.
[14] R.L. Graham, D.E. Knuth, T.S. Motzkin, Complements and transitive closures, Discrete Math. 2 (1972) 17–29.
[15] B. Hansson, Choice structures and preference relations, Synthese 18 (1968) 443–458.
[16] C. Kuratowski, Sur l'opération $\bar{A}$ de l'analysis situs, Fund. Math. 3 (1922) 182–199.
[17] D. Peleg, A generalized closure and complement phenomenon, Discrete Math. 50 (1984) 285–293.
[18] H. Raiffa, Decision Analysis, Addison-Wesley, Reading, Mass, 1968.
[19] K. Suzumura, Remarks on the theory of collective choice, Economica 43 (1976) 381–390.
[20] K. Suzumura, Rational Choice, Collective Decisions, and Social Welfare, Cambridge University Press, Cambridge, 1983.
[21] E. Szpilrajn, Sur l'extension de l'ordre partiel, Fund. Math. 16 (1930) 386–389.
[22] A. Tversky, Intransitivity of preferences, Psychol. Rev. 76 (1969) 31–48.